Holographic Renormalization and Stress Tensors in New Massive Gravity

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Abstract

We obtain holographically renormalized boundary stress tensor with the emphasis on a special point in the parameter space of three dimensional new massive gravity, using the so-called Fefferman-Graham coordinates with relevant counter terms. Through the linearized equations of motion with a standard prescription, we also obtain correlators among these stress tensors. We argue that the self-consistency of holographic renormalization determines counter terms up to unphysical ambiguities. Using this renormalized stress tensor in Fefferman-Graham coordinates, we obtain the central charges of dual CFT, and mass and angular momentum of some AdS black hole solutions. These results are consistent with the previous ones obtained by other methods. In this study on the Fefferman-Graham expansion of new massive gravity, some aspects of higher curvature gravity are revealed.

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1 Introduction

It is natural to consider higher curvature corrections in gravity theories beyond an Einstein-Hilbert term with(out) a cosmological constant in the viewpoint of string theory or quantum gravity, though there are some attempts, for instance, loop quantum gravity approach, to make pure Einstein gravity into self-consistent quantum theory without such corrections. However, it is not easy task to perform a detailed analysis of higher curvature gravity in more than four dimensions, partly because of the complicated dynamics and the difficulty in obtaining analytic solutions. Moreover, the higher derivative terms with general covariance may conflict with unitarity in many cases. The situation is, more or less, different in lower dimensional higher curvature gravity theories since the reduction in degrees of freedom simplifies the dynamics and the existence of analytic black hole solutions leads to some analytic results. In particular, the three dimensional higher curvature theory has non-empty contents and it may admit non-trivial black hole solutions of AdS asymptotics. Therefore, it is useful route to study three dimensional gravity theory to obtain insights on gravity with higher derivative terms in conjunction with AdS/CFT correspondence [1, 2]. It is also notable that the reliable quantum computation in gravity without supersymmetry is performed only in three dimensions [3] and that Kerr/CFT [4] is inspired by AdS/CFT correspondence on three dimensions.

In last few years, the specific higher curvature gravity theory in three dimensions, now known as new massive gravity (NMG) [5, 6, 7, 8], has drawn some interests and leads to another realization of the AdS/CFT correspondence. The original motivation of NMG is the non-linear completion of the Fierz-Pauli massive graviton theory, which is not yet done in higher dimensions. Concretely, the simplest form of NMG is composed of a curvature scalar term with(out) a cosmological constant and the other term which is the specific combination of curvature scalar square and Ricci tensor square. This parity-even NMG shares some aspects with the so-called topologically massive gravity (TMG) [9, 10] in three dimensions, which is composed of a curvature scalar term with a cosmological constant and a gravitational Chern-Simons term. One of common aspects in these two theories is the existence of AdS 'vacuum' solution. Soon after the introduction of NMG, it was realized that AdS space may be allowed as a solution, and so some studies [11, 12, 13, 14, 15, 16] are done along the route of AdS/CFT correspondence in this new setup, similarly to the case in TMG [17, 18, 19, 20]. Basically, all these studies do not require the supersymmetry as a crucial ingredients and so may be regarded as a test or the realization of AdS/CFT correspondence just with bosonic degrees of freedom.

Aside from some similarity to TMG, NMG has several different aspects from TMG. Because of these, various methods developed for TMG may not be applied directly to NMG. One peculiar thing in NMG is that the radius of the AdS space is not identical with a cosmological constant, since the higher curvature term may also play the role of an effective cosmological constant. Another notable difference resides in the fact that there is no guarantee of the completeness of NMG from the very beginning. On the contrary, there are some suggestions of the completeness of TMG at a certain special point in the parameter space [18], though there are several arguments against these [21]. It turns out that this incompleteness in NMG is not just drawbacks but allows the extension of NMG to even higher curvature gravity theories [22, 23]. The underlying principle in this extension is again the renowned AdS/CFT correspondence. The important observation in this development is the compliance of the NMG with the holographic c-theorem [22, 24], which is a specific incarnation of AdS/CFT correspondence in this context. The consistency with the holographic c-theorem might

be an important ingredient for quantum gravity if we suppose that AdS/CFT correspondence has a deep meaning in quantum gravity. Because the simplest form of NMG is uniquely determined by the holographic c-theorem, it is a proper step to study various aspects of NMG in this simplest case. In this paper, we will focus on the simplest form of NMG, and will call this form just as NMG in the following.

Though a primitive form of AdS/CFT correspondence was found in three dimensions by Brown and Henneaux [25], the concrete version of AdS/CFT correspondence from string theory was realized in Maldacena's work [1] and afterwards the prescription for the matching between partition functions are established [2]. Specifically, on-shell bulk gravity action on AdS space becomes a functional of boundary values after solving equations of motion (EOM) of the bulk modes in terms of the boundary values. Then, the boundary values of non-normalizable modes of bulk gravity theory play the role of sources of corresponding operators in the dual CFT side, and one can evaluate (at least, large 't Hooft coupling) correlators of the corresponding operators through gravity computations. In particular, the boundary stress tensor in CFT couple to the boundary metric which is naturally identified with the induced metric from the bulk one. Therefore, those can be obtained just from the pure gravity modes. As usual in other operators in field theories, stress tensor requires some renormalization for finite results. This process is also realized in the side of bulk gravity through AdS/CFT correspondence, which is now coined as *holographic renormalization* [26, 27, 28, 29, 30, 31].

In a gravity theory with a boundary, one may need a boundary term in the action to ensure the well-defined variational principle. In the case of pure Einstein gravity, this boundary term has been called the Gibbons-Hawking (GH) term [32], which turns out to be just the extrinsic curvature of the boundary. With the equipment of this GH term boundary stress tensor was introduced a long time ago by Brown and York [33], which contains divergence in the case of AdS space. According to AdS/CFT dictionary, this fact has been interpreted as the counter part of the necessity of the renormalization in the boundary CFT, and so some counter terms, which are local in the boundary and consistent with the variational principle, are constructed and lead to finite results for boundary stress tensor. This holographic renormalization is one of the highly successful realization of AdS/CFT correspondence. Though this construction is conceptually transparent, it has been not yet implemented for most of higher curvature gravity theories since there are several challenging steps to construct holographically renormalized boundary stress tensor. For instance, a relevant GH term has not yet been found for most of higher derivative gravity theories. However, one may note that holographic renormalization was implemented successfully in TMG without information about GH term [34].

In NMG the generalized GH term was obtained with the aim of AdS/CFT correspondence [35] and holographically renormalized stress tensor was also studied for mass and angular momentum of some AdS black holes [36]. Furthermore, some correlators of renormalized stress tensor are obtained at the so-called *critical point*, revealing characteristic of logarithmic CFT (LCFT) from bulk log modes [37, 38, 39]. Even with these studies there are some missing parts in the analysis and it is also desirable to study holographic renormalization in the unified manner. One may note that there is another special point other than critical point in NMG, which is related to the existence of the so-called *new type black holes* [6, 12, 40, 41, 42]. At this special point, it has been known that a different fall-off of bulk modes is allowed. While mass and angular momentum of new type black holes are identified as conserved charges [40] and consistent with renormalized stress

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tensor [36], the central charge of dual CFT in this case has been obtained indirectly through Cardy formula [12] or central charge function formalism [43]. According to holographic renormalization or AdS/CFT correspondence, all these quantities should be obtained uniformly from renormalized boundary stress tensor.

In this paper, we study the holographic renormalization of boundary stress tensor in the so-called Fefferman-Graham coordinates [44] which are especially suitable to the purpose and relevant to obtaining several physical quantities in the unified manner. We argue that the consistency of holographic renormalization determines the relevant counter terms up to unphysical ambiguities. This is one of improvements over the previous related studies. The organization of the paper is as follows. In the next section Fefferman-Graham coordinates are adopted and Brown-York boundary tensor is obtained. In section 3 several possible counter terms are considered and the corresponding renormalized boundary stress tensor is obtained. Accordingly, the central charge of dual CFT is obtained as a trace anomaly of renormalized stress tensor. Using linearized EOMs in NMG with the standard prescription in the AdS/CFT correspondence, we obtain the correlators of boundary stress tensor in section 4, which give us the central charge of dual CFT and some other information. In this section we argue that the consistency of holographic renormalization determines counter terms up to unphysical ambiguities. In section 5, through renormalized stress tensor dictated by holographic renormalization we obtain mass and angular momentum of some AdS black holes, which are consistent with previous results obtained by other methods. In the final section we summarize our results with some comments. In appendixes, useful formulae are collected, which are used in the main text.

2 Fefferman-Graham Expansion in NMG

After introduced as a completion of Fierz-Pauli linear graviton theory, NMG was recognized as a consistent form with holographic c-theorem and was in fact uniquely determined by it. This uniqueness of Lagrangian provides a strong support for AdS/CFT correspondence of any kind of AdS solutions in this theory. Therefore, it is desirable to analyze the boundary stress tensor for AdS asymptotics in the unified manner for any parameter values in the Lagrangian. For this purpose it turns out that the so-called Fefferman-Graham coordinates are appropriate ones. The Lagrangian of NMG we will consider in the following is given by

$$S = \frac{\xi}{2\kappa^2} \int d^3x \sqrt{-g} \left[\sigma R + \frac{2}{\ell^2} + \frac{1}{m^2} \mathcal{K} \right], \qquad (1)$$

where ξ and σ take 1 or -1, $2\kappa^2 = 16\pi G$, and $\mathcal{K}$ is a specific combination of scalar curvature square and Ricci tensor square defined by

$$\mathcal{K} = R_{\mu\nu}R^{\mu\nu} - \frac{3}{8}R^2. \qquad (2)$$

Our convention is such that m^2 is always positive but the cosmological constant ℓ^2 has no such restriction[1]. The equations of motion of NMG are given by

$$\mathcal{E}_{\mu\nu} = \xi \left[\sigma G_{\mu\nu} - \frac{1}{\ell^2} g_{\mu\nu} + \frac{1}{2m^2} \mathcal{K}_{\mu\nu} \right] = 0, \qquad (3)$$

---

[1]We have introduced ξ for the various sign choice of terms in the action. but it will be set unity in the following.

where

$$\mathcal{K}_{\mu\nu} = g_{\mu\nu}\left(3R_{\alpha\beta}R^{\alpha\beta} - \frac{13}{8}R^2\right) + \frac{9}{2}RR_{\mu\nu} - 8R_{\mu\alpha}R^\alpha_\nu + \frac{1}{2}\left(4\mathcal{D}^2R_{\mu\nu} - \mathcal{D}_\mu\mathcal{D}_\nu R - g_{\mu\nu}\mathcal{D}^2R\right). \quad (4)$$

In the above Lagrangian of NMG there is an additional parameter m^2 of mass dimension two in front of $\mathcal{K}$ term along with the gravitational constant G and cosmological constant ℓ. In NMG AdS space is allowed as a solution and its radius L, which is always positive, is related to parameters of the Lagrangian as

$$\frac{1}{L^2} = 2m^2\left[\sigma \pm \sqrt{1 - \frac{1}{m^2\ell^2}}\right]. \quad (5)$$

Instead of a cosmological constant ℓ, we will use L as a basic parameter in the following since we will confine ourselves to the case of asymptotically AdS space. Note that the dimensionless combinations of three parameters can be represented by L/G and m^2L^2 which are positive quantities.

Because of the additional parameter m^2, there are several special points in the parameter space, one of which is now coined as *critical point* [45, 46, 47]. At this point which is given by the parameter condition $-\sigma = 2m^2L^2 = 1$ ($m^2\ell^2 = -1/3$) in our convention, it was shown that there may be log modes in the bulk gravity which corresponds to the non-unitary LCFT at the boundary. These log tails of bulk modes differ from Brown-Henneaux fall-off boundary condition and therefore the correspondence at this point is regarded as the extension of the standard AdS/CFT correspondence. Another interesting point of parameter space is given by $\sigma = 2m^2L^2 = 1$ ($m^2\ell^2 = 1$). At this special point, there are AdS black hole solutions which have different fall-off behaviors from Brown-Teitelboim-Zanelli (BTZ) ones [48]. As in the case of the above critical point, it is reasonable to perform further analysis at this special point in the viewpoint of holographic renormalization, which is the main focus in this paper.

Through the AdS/CFT correspondence, holographically renormalized boundary stress tensor corresponds to one point function of stress tensor in the dual CFT, and so can be used to obtain the central charge of the dual CFT. It has been known that those can also be used to obtain mass and angular momentum of relevant AdS black holes. The holographically renormalized stress tensor may be introduced in the following way. First, let us obtain the Brown-York boundary stress tensor for which one needs the GH boundary term for the well-defined variational procedure. This Brown-York stress tensor is usually divergent because of the nature of asymptotic AdS space. And then, by a suitable choice of local counter terms in the boundary, one can introduce the renormalized stress tensor. To apply this procedure with AdS asymptotics, it has been known that the Fefferman-Graham coordinates are especially appropriate.

Now, let us briefly explain the Fefferman-Graham expansion in our convention. Asymptotically AdS space may be put in the following metric form which is useful for our purpose

$$ds^2 = L^2\gamma_{\mu\nu}dx^\mu dx^\nu = L^2\left[d\eta^2 + \gamma_{ij}dx^i dx^j\right],$$

$$(6)$$

$$\gamma_{ij} = e^{2\eta}g^{(0)}_{ij} + e^\eta g^{(1)}_{ij} + g^{(2)}_{ij} + e^{-\eta}g^{(3)}_{ij} + e^{-2\eta}g^{(4)}_{ij} + \mathcal{O}(e^{-3\eta}).$$

Non-vanishing Christoffel symbols in the above Fefferman-Graham coordinates are given by

$$\Gamma^\eta_{ij} = -K_{ij}, \qquad \Gamma^i_{\eta j} = K^i_j, \qquad \Gamma^k_{ij} = {}^{(2)}\Gamma^k_{ij}(\gamma),$$

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where K_{ij} is the extrinsic curvature tensor given by $K_{ij} = \frac{1}{2}\partial_\eta\gamma_{ij}$ in this coordinate system and $^{(2)}\Gamma^k_{ij}$ is two dimensional Christoffel symbols given by the metric γ. In the following $g_{(0)}$ is taken as the boundary metric for two dimensional CFT and geometrical quantities at the boundary are denoted with the script (0). For instance, the boundary scalar curvature is denoted as $R_{(0)}$. In these coordinates Ricci tensor is given by

$$
\begin{aligned}
R^\eta{}_\eta &= -K' - K_{ij}K^{ij}\,, \\
R^\eta{}_i &= \nabla^j K_{ji} - \nabla_i K\,, \\
R^i{}_j &= {}^{(2)}R^i{}_j - (K^i_j)' - KK^i{}_j\,, \\
R &= {}^{(2)}R - K_{ij}K^{ij} - K^2 - 2K'\,,
\end{aligned}
\tag{7}
$$

where $'$ denotes the differentiation with respect to η, $K \equiv \gamma^{ij}K_{ij}$, $^{(2)}R^i{}_j$ is the two dimensional Ricci tensor, and ∇ denotes the covariant derivative with respect to the two dimensional metric γ.

The above coordinates are known as Fefferman-Graham coordinates (or expansion) and extremely useful for holographic renormalization. In the following, the fluctuation modes as well as background space are taken in these coordinates and the AdS radius is set as the unity for the convenience, *i.e.* $L = 1$ which can be reinstated by a simple dimensional reasoning. One may notice that we have included $g_{(k)}$ terms of odd k deliberately in (6), which turns out to be related to higher curvature effects. In the case of usual pure Einstein gravity and TMG, $g_{(k)}$ terms of odd k have been dropped because they vanish by EOM. Contrary to these, we will show that this is not the case at the special parameter point in NMG. Odd k terms are not taken in previous analysis [37] in NMG since a critical point not our special point is the main interest. One may note that the relevance of $g_{(k)}$ terms of odd k is mentioned in a certain matter-coupled three dimensional gravity without a detailed analysis [34, 49].

For completeness, let us briefly review holographic renormalization and renormalized stress tensor in the case of pure Einstein-Hilbert gravity with the focus on the central charge of dual CFT. The boundary renormalized stress tensor, T^{ij} is introduced as

$$
\delta(S_{Bulk} + S_{GH} + S_{c.t.}) = \frac{1}{2}\int d^2x \sqrt{-g_{(0)}}\, T^{ij}\delta g^{(0)}_{ij}\,,
\tag{8}
$$

where S_{Bulk} is a bulk gravity action, S_{GH} is a GH boundary term and $S_{c.t.}$ is the so-called counter term which renders finite the boundary stress tensor. Two terms S_{Bulk} and S_{GH} give the Brown-York boundary stress tensor and the counter term does a counter boundary stress tensor. In the above Fefferman-Graham coordinates one can see that the renormalized boundary stress tensor can be obtained by the following formula

$$
8\pi G \sqrt{-g_{(0)}}\, T^{ij} = 8\pi G e^{2\eta}\sqrt{-\gamma}\Big[T^{ij}_{BY} + T^{ij}_{c.t.}\Big]_{\eta\to\infty}\,,
\tag{9}
$$

where T^{ij}_{BY} is Brown-York boundary stress tensor and $T^{ij}_{c.t.}$ is counter one. One may note that the divergences of stress tensor without a counter term come from the combined quantity $e^{2\eta}\sqrt{-\gamma}\, T_{BY}$ in our convention. For brevity we call these divergences as those of Brown-York tensor in the following.

In the pure Einstein gravity case almost unique choice of counter term, which is local at the

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boundary, is just the boundary cosmological constant with a suitable coefficient[2] [31, 50]

$$S_{c.t.} = -\frac{1}{8\pi G} \int d^2x \sqrt{-\gamma} \, .$$

In this case Brown-York and counter stress tensor are given by

$$8\pi G\, T_{BY}^{ij} = K\gamma^{ij} - K^{ij} \, , \qquad 8\pi G\, T_{c.t.}^{ij} = -\gamma^{ij} \, .$$

Then, the renormalized stress tensor is given explicitly by

$$8\pi G \sqrt{-g_{(0)}}\, T^{ij} = e^{2\eta}\sqrt{-\gamma}\Big[(K-1)\gamma^{ij} - K^{ij}\Big]_{\eta\to\infty} \, , \tag{10}$$

which lead to

$$8\pi G\, T^{ij} = g_{(2)}^{ij} - (\text{Tr}\, g_{(2)})g_{(0)}^{ij} \, .$$

In this expression all quantities are raised or lowered by the boundary metric $g_{(0)}$ and 'Tr' denotes the contraction by $g_{(0)}$. Note that $g_{(1)} = 0$ and $R_{(0)} + 2\text{Tr}\, g_{(2)} = 0$ from the EOMs. One can see that these give us the correct mass and angular momentum of BTZ black holes and the central charge of dual CFT. Concretely for the central charge, one can see that the contraction of renormalized stress tensor is given by [31, 50]

$$8\pi G\, T = -\text{Tr}\, g_{(2)} \, .$$

Using EOM in Fefferman-Graham coordinates and recalling $\langle T \rangle = (c/24\pi)R_{(0)}$ in two dimensional conformal anomaly through the identification of boundary CFT stress tensor with holographic one, one obtains the central charge of dual CFT as $c = 3/2G$.

Now let us recall the generalized GH term in NMG. To obtain the generalized GH term in NMG, Hohm and Tonni has used the auxiliary field approach[3] [35]. In summary, the above NMG action can be rewritten in terms of auxiliary field $f_{\mu\nu}$ as

$$S = \frac{1}{2\kappa^2} \int d^3x \sqrt{-g}\Big[\sigma R + \frac{2}{\ell^2} + f^{\mu\nu}G_{\mu\nu} - \frac{m^2}{4}\Big(f^{\mu\nu}f_{\mu\nu} - f^2\Big)\Big] \, . \tag{11}$$

In this representation the EOMs of this action are given by

$$\sigma G_{\mu\nu} - \frac{1}{\ell^2}g_{\mu\nu} = \mathcal{T}_{\mu\nu}^B \, , \qquad f_{\mu\nu} = \frac{2}{m^2}\Big(R_{\mu\nu} - \frac{1}{4}Rg_{\mu\nu}\Big) \, , \tag{12}$$

where

$$\begin{aligned}
\mathcal{T}_{\mu\nu}^B &= \frac{m^2}{2}\Big[f_{\mu\alpha}f_\nu^\alpha - ff_{\mu\nu} - \frac{1}{4}\Big(f^{\alpha\beta}f_{\alpha\beta} - f^2\Big)g_{\mu\nu}\Big] + \frac{1}{2}fR_{\mu\nu} - \frac{1}{2}Rf_{\mu\nu} - 2f_{\alpha(\mu}G_{\nu)}^\alpha + \frac{1}{2}f^{\alpha\beta}G_{\alpha\beta}g_{\mu\nu} \\
&\quad - \frac{1}{2}\Big[\mathcal{D}^2f_{\mu\nu} + \mathcal{D}_\mu\mathcal{D}_\nu f - 2\mathcal{D}^\alpha\mathcal{D}_{(\mu}f_{\nu)\alpha} + \Big(\mathcal{D}_\alpha\mathcal{D}_\beta f^{\alpha\beta} - \mathcal{D}^2 f\Big)g_{\mu\nu}\Big] \, .
\end{aligned}$$

This form of NMG is useful to obtain the generalized Gibbons-Hawking boundary term. As the metric decomposition, auxiliary fields $f^{\mu\nu}$ can be decomposed as

$$f^{\mu\nu} = \begin{pmatrix} s & h^j \\ h^i & f^{ij} \end{pmatrix} \, .$$

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[2]It is well known that there should be another term of the form $(1/16\pi G)\eta \int d^2x\sqrt{-\gamma}R$, to render on-shell action finite. However, this term is topological in two dimensions and so irrelevant to stress tensor computation in the following. See Appendix D.

[3]One may note that our convention is slightly different from [35].

With this decomposition the generalized GH term was obtained in the form of

$$S_{GH} = \frac{1}{2\kappa^2} \int d^2x \sqrt{-\gamma} \Big[2\sigma K + \hat{f}^{ij} K_{ij} - \hat{f} K \Big] , \qquad (13)$$

where the first term proportional to σ is the GH term in pure Einstein gravity case and $\hat{f}^{ij}$, $\hat{f}$ are given in [35]. With this result, Brown-York stress tensor for NMG is also obtained in [35, 36]. After some rearrangement those can be written as

$$8\pi G T_{BY}^{ij} = \Big(\sigma + \frac{1}{2}\hat{s} - \frac{1}{2}\hat{f} \Big)(K\gamma^{ij} - K^{ij}) - \nabla^{(i}\hat{h}^{j)} + \frac{1}{2}D_\eta \hat{f}^{ij} + K_k^{(i}\hat{f}^{j)k} + \gamma^{ij}\Big(\nabla_k \hat{h}^k - \frac{1}{2}D_\eta \hat{f} \Big),$$

where the first term proportional to σ is just the Brown-York tensor for the pure gravity. Note that in the Fefferman-Graham coordinate system D_η becomes the ordinary derivative with respect to η. That is to say, $D_r = \partial_\eta$ and $\hat{}$ has no effect in our case: $\hat{f} = \gamma_{ij}f^{ij}$, $\hat{h} = h$, $\hat{s} = s$ and $f = \gamma_{\mu\nu}f^{\mu\nu} = \hat{f} + s$. It is important to remind that f_j^i, neither f_{ij} nor f^{ij}, are taken as fundamental variables to construct generalized GH term and accordingly Brown-York stress tensor.

One can now see that Brown-York tensor, T_{BY}^{ij} expanded in Fefferman-Graham coordinates is given by

$$
\begin{aligned}
8\pi G T_{BY}^{ij} =\ & e^{-2\eta}\Big(\sigma + \frac{1}{2m^2} \Big) g_{(0)}^{ij} - \frac{1}{2} e^{-3\eta} \Big[\Big(\sigma + \frac{1}{2m^2} \Big)(\mathrm{Tr}\, g_{(1)})\, g_{(0)}^{ij} + \Big(\sigma + \frac{3}{2m^2} \Big) g_{(1)}^{ij} \Big] \\
& - e^{-4\eta} \Big[\Big\{ \frac{1}{4m^2}R_{(0)} + \Big(\sigma + \frac{1}{m^2} \Big)\mathrm{Tr}\, g_{(2)} - \frac{1}{2}\Big(\sigma + \frac{5}{8m^2} \Big)\mathrm{Tr}\, g_{(1)}^2 - \frac{1}{16m^2}(\mathrm{Tr}\, g_{(1)})^2 \Big\} g_{(0)}^{ij} \\
& - \frac{1}{2m^2}g_{(1)}^{ik}g_{(1)\,k}^{j} - \frac{1}{2}\Big(\sigma + \frac{1}{m^2} \Big)(\mathrm{Tr}\, g_{(1)})\, g_{(1)}^{ij} \Big] + \mathcal{O}(e^{-5\eta}) . \qquad (14)
\end{aligned}
$$

Note that there are two divergences which come from the first and the second terms in the above Brown-York tensor. Though there are some proposals which counter term is appropriate for the renormalized boundary stress tensor in the case of NMG, which will be reviewed in the next section, it is desirable to investigate counter terms and renormalized stress tensor in the viewpoint of holographic renormalization with Fefferman-Graham coordinates. It turns out that some features are transparent in Fefferman-Graham coordinates.

3   Counter Terms and Renormalized Stress Tensors

In this section we construct relevant counter terms in NMG and then obtain renormalized stress tensor under the perspective of holographic renormalization. This stress tensor gives us the correct central charge of dual CFT and mass and angular momentum of AdS black holes, which are consistent with previous results obtained by other methods. There are some studies about holographically renormalized stress tensor in some parameter regions, which are briefly reviewed in the following. Though renormalized stress tensor at the special point of $\sigma = 2m^2 = 1$ was studied [36], there are some gaps in the logic and not sufficiently generic. For instance, the central charge of dual CFT is dealt with differently from mass and angular momentum of AdS black holes. As mentioned in the introduction, all these quantities can be managed at one stroke in Fefferman-Graham coordinates. Furthermore, this approach does not resort on a particular black hole solutions rather uses

just the generic expansion form. In addition to this improvement, Fefferman-Graham expansion is very useful to obtain correlators of stress tensor through holographic renormalization, which are presented in the next section.

There have been two suggestions for relevant counter terms in NMG [35, 36]. In our perspective, the first case corresponds to $g_{(1)} = 0$ and the second one does to $g_{(1)} \neq 0$. Now, let us explain these choices of counter terms in the viewpoint of our approach. The first prescription for counter term is given in [35] for the so-called Brown-Henneaux fall-off boundary conditions. The relevant counter term is taken by the boundary cosmological constant just like the pure gravity case with a suitable coefficient

$$S_{c.t.} = -\Big(\sigma + \frac{1}{2m^2}\Big)\frac{1}{8\pi G}\int d^2x\sqrt{-\gamma}\,.$$

In this choice of a counter term, the leading divergence of the Brown-York tensor can be canceled, which is the unique divergent term under the condition of $g_{(1)} = 0$. It was also shown that renormalized stress tensor reproduces the correct central charge of dual CFT using the asymptotic symmetry algebra, which is consistent with the results by central charge function formula or by Cardy formula. However, it is insufficient for the $g_{(1)} \neq 0$ case as one can see the expression of Brown-York tensor in Eq. (14). There is another divergence coming from the $g_{(1)}$ part. This may be paraphrased as this counter term is appropriate to Brown-Henneaux boundary conditions but it is insufficient for weaker boundary conditions than Brown-Henneaux.

A different counter term for weaker fall-off boundary condition than Brown-Henneaux one is taken in [36] as

$$S_{c.t.} = \frac{m^2}{2}\Big(\sigma + \frac{1}{2m^2}\Big)\frac{1}{8\pi G}\int d^2x\sqrt{-\gamma}\,\hat{f}\,.$$

At first sight, one may suspect that the chosen counter term cancels only the leading divergence and a part of the next leading one of Brown-York tensor, not the whole of the next leading one. However, one can see that the remnant potential divergence cancels completely, either since $g_{(1)}$ term vanishes by EOM at the generic parameter point or since the coefficient of next leading divergence vanishes at the special point of $\sigma = 2m^2 = 1$. Though the above choice of counter term cancels all the divergences in NMG at the special parameter point, its inference is neither completely logical nor unique since it depends on specific black hole solutions called new type black holes. Moreover, it was argued that one needs to consider rotating new type black holes for the complete determination.

Meanwhile, in holographic renormalization black hole solutions are not essential to obtain central charges, as can be seen from the fact that the appearance of log modes at the critical point cannot be deduced from known black hole solutions [38, 39]. To overcome this logical drawback and clarify the meaning of freedom in counter term choice, let us consider generic counter terms without resorting to specific black hole solutions. Then, it is legitimate to consider the following generic counter terms

$$S_{c.t.} = \frac{1}{8\pi G}\int d^2x\sqrt{-\gamma}\,(A + B\hat{f} + C\hat{f}^2 + Df_{ij}f^{ij})\,,$$

which lead to

$$8\pi G\,T_{c.t.}^{ij} = (A + B\hat{f} + C\hat{f}^2 + Df_{kl}f^{kl})\,\gamma^{ij}\,.$$

To obtain this expression, it is crucial to recall that the fundamental fields under the variation are f^μ_ν neither $f^{\mu\nu}$ nor $f_{\mu\nu}$, as advertised in [35] .

To cancel the divergence, one should take A, B, C, D as

$$0 \;=\; A - \frac{2}{m^2}B + \frac{4}{m^4}C + \frac{2}{m^4}D + \sigma + \frac{1}{2m^2}\,, \tag{15}$$

$$0 \;=\; \frac{1}{m^2}B - \frac{4}{m^4}C - \frac{2}{m^4}D - \frac{1}{2}\Big(\sigma + \frac{1}{2m^2}\Big)\,. \tag{16}$$

The first condition is needed for the cancelation of the leading divergence in Brown-York tensor, and the second one for the cancelation of the next leading one. One may note that the second condition is not necessary for the case of $g_{(1)} = 0$, which is implied by EOMs at the generic point in the parameter space. In other words, there are ambiguities in choosing counter terms at the generic point in the parameter space. One can see that any choice of A, B, C, D satisfying the first condition gives us the same results for on-shell boundary stress tensor. To see this, note that the renormalized boundary stress tensor is given by

$$8\pi G\, T^{ij} \;=\; \Big(\sigma + \frac{1}{2m^2}\Big)g_{(2)}{}^{ij} - \Big[\frac{1}{4m^2}R_{(0)} + \Big(\sigma + \frac{1}{m^2}\Big)\mathrm{Tr}\, g_{(2)}$$
$$- \Big(\frac{1}{m^2}B - \frac{4}{m^4}C - \frac{2}{m^4}D\Big)\Big(R_{(0)} + 2\mathrm{Tr}\, g_{(2)}\Big)\Big]g_{(0)}^{ij}\,. \tag{17}$$

To impose on-shell condition, let us note that $\eta\eta$-components of EOMs or the contracted one of EOMs up to relevant orders lead to

$$R_{(0)} + 2\mathrm{Tr}\, g_{(2)} = 0\,. \tag{18}$$

Since EOMs in NMG are complicated by higher curvature terms, some details about EOMs are relegated to appendixes. See there for the derivation of the above. As a result, one obtains

$$8\pi G\, T^{ij} = \Big(\sigma + \frac{1}{2m^2}\Big)\Big[g_{(2)}{}^{ij} - (\mathrm{Tr}\, g_{(2)})g_{(0)}{}^{ij}\Big]\,, \tag{19}$$

which shows us ambiguities in counter terms in this case. This phenomenon is a reminiscent of the renormalization scheme independence in dual field theory. The above renormalized stress tensor under the contraction with the boundary metric $g_{(0)}$ leads to

$$8\pi G\, T = \frac{1}{2}\Big(\sigma + \frac{1}{2m^2}\Big)R_{(0)}\,, \tag{20}$$

and then central charge is obtained as

$$c = \Big(\sigma + \frac{1}{2m^2}\Big)\frac{3}{2G}\,, \tag{21}$$

which reproduces the results in [35, 51]. This shows us that the boundary cosmological constant as a counter term is not unique choice but may be regarded as a minimal one.

In the case of $g_{(1)} \neq 0$, which is allowed only at the point $\sigma = 2m^2 = 1$ for the finiteness of the stress tensor (22), we should impose the second condition. Even in this case, there is still ambiguity in choosing the counter terms as can be seen explicitly as follows. Firstly, under the above two conditions the renormalized boundary stress tensor is given by

$$8\pi G\, T^{ij} \;=\; \frac{1}{2}e^{\eta}\Big(\sigma - \frac{1}{2m^2}\Big)g_{(1)}^{ij} + \Big(\sigma + \frac{1}{2m^2}\Big)g_{(2)}^{ij} - \sigma g_{(1)}{}^i{}_k g_{(1)}^{kj} + \frac{1}{4}\Big(\sigma + \frac{1}{2m^2}\Big)(\mathrm{Tr}\, g_{(1)})g_{(1)}^{ij}$$
$$+ \Big[\frac{1}{2}\sigma R_{(0)} - \frac{1}{2m^2}\mathrm{Tr}\, g_{(2)} + C_1\mathrm{Tr}\, g_{(1)}^2 + C_2(\mathrm{Tr}\, g_{(1)})^2\Big]g_{(0)}^{ij}\,, \tag{22}$$

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where
$$C_1 = -\frac{3}{4m^2}B + \frac{3}{m^4}C + \frac{5}{2m^4}D + \frac{1}{2}\left(\sigma + \frac{5}{8m^2}\right), \qquad C_2 = -\frac{1}{4m^2}B + \frac{2}{m^4}C + \frac{1}{2m^4}D + \frac{1}{16m^2}.$$

After the conditions for the cancelation of divergences are imposed, these are given by

$$C_1 = \frac{1}{m^4}D + \frac{1}{8}\left(\sigma + \frac{1}{m^2}\right), \qquad C_2 = \frac{1}{m^4}C - \frac{1}{8}\sigma.$$

Note that the $\eta\eta$-component of EOMs in this case, $\mathcal{E}^\eta{}_\eta = 0$ gives us the following equations (See Appendix B)

$$\mathrm{Tr}\, g_{(1)}^2 = (\mathrm{Tr}\, g_{(1)})^2, \qquad R_{(0)} + 2\mathrm{Tr}\, g_{(2)} - \frac{1}{2}(\mathrm{Tr}\, g_{(1)})^2 = 0. \tag{23}$$

Using the first equation in the above, one obtains at the special point of $\sigma = 2m^2 = 1$

$$8\pi G T^{ij} = 2g_{(2)}^{ij} - g_{(1)}{}^i{}_k g_{(1)}^{kj} + \frac{1}{2}(\mathrm{Tr}\, g_{(1)})g_{(1)}^{ij} + \left[\frac{1}{2}R_{(0)} - \mathrm{Tr}\, g_{(2)} + \left\{4(C+D) + \frac{1}{4}\right\}(\mathrm{Tr}\, g_{(1)})^2\right]g_{(0)}^{ij}. \tag{24}$$

As can be seen in the above, one needs another condition to determine renormalized stress tensor unambiguously. In fact, by supposing the validity of AdS/CFT correspondence in this case one can see that there should be another condition given by

$$C + D = 0. \tag{25}$$

This condition is accounted for by supposing the consistency of holographic renormalization through correlator computation among boundary stress tensors, which will be done in section 4. Supposing this condition to hold, *on-shell* holographically renormalized boundary stress tensor is independent of counter terms and then the trace part of the renormalized boundary stress tensor is, through the second equation in Eq. (23), given by

$$8\pi G T = R_{(0)}. \tag{26}$$

As a result, one can see that the central charge of dual CFT in this case is given by

$$c = \frac{3}{G}, \tag{27}$$

which is twice of the pure Einstein gravity case and consistent with Cardy formula [12] or slightly modified central charge function formalism [43].

One may wonder whether it is possible to consider the following more general form of counter terms

$$S_{c.t.} = \frac{1}{8\pi G}\int d^2x\sqrt{-\gamma}\,(A + B\hat{f} + C\hat{f}^2 + Df_{ij}f^{ij} + Es + Fs^2).$$

As one can see from EOMs, s terms lead just to the constant value when EOM's are imposed, which plays the same role with the boundary cosmological constant. Therefore, it cannot be used as new counter terms at all. One may also wonder the possibility of adding even higher order terms in $\hat{f}$ or f_{ij}. These do not lead to further possibilities, since these give us the unphysical ambiguities up to the relevant order in boundary stress tensor. Some more comments are in order. Our results imply that the cancelation of divergences are not sufficient to determine the relevant counter terms completely, which was the case of the pure Einstein gravity. Even with some additional information from the compliance with AdS/CFT correspondence, which will be done in section 4, there is still ambiguity in counter terms, though this ambiguity does not affect physical quantities like central charges.

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4 Linearized Analysis and Stress Tensor Correlators

The linearized analysis in the Fefferman-Graham coordinates is done in NMG at the so-called critical point $-\sigma = 2m^2 = 1$, where the existence of log modes leads to non-unitary boundary CFT known as LCFT [38, 39] . In our main interest, the values of parameters are different from the critical point and the analysis should be done independently. In this section we present the linearized expressions of some quantities and then represent stress tensor in terms of boundary values of linearized metric in generic parameter values as well as in the special point $\sigma = 2m^2 = 1$. Then, we obtain correlators of stress tensor, according to the standard AdS/CFT dictionary.

One may note that Fefferman-Graham expansion contains sufficient information for the linearized analysis, in some sense. Once Fefferman-Graham expansion is obtained, the linearized analysis can be implemented easily. For the linearized analysis the background AdS space can be taken just by the flat boundary metric, η_{ij} with the radial part. Then, the $n(\geq 1)$ order parts in Fefferman-Graham expansion of two dimensional metric, γ, can be regarded as the small fluctuation ones. In addition, the boundary metric $g_{(0)}$ is also decomposed as the flat one and the small fluctuation part $h_{(0)}$. In summary, one may take

$$g_{ij}^{(0)} = \eta_{ij} + h_{ij}^{(0)} \,, \qquad g_{ij}^{(n)} = h_{ij}^{(n)} \,, \tag{28}$$

and ignore higher order terms for h in the Fefferman-Graham expanded expressions.

As a preliminary step, let us consider the pure Einstein gravity case with negative cosmological constant. Renormalized boundary stress tensor in terms of linearized metric can be obtained from the Fefferman-Graham expanded form as

$$8\pi G\,T^{ij} = g_{(2)}^{ij} - (\text{Tr}\,g_{(2)})g_{(0)}^{ij} = g_{(2)}^{ij} + \frac{1}{2}R_{(0)}g_{(0)}^{ij} = h_{(2)}^{ij} + \frac{1}{2}(\partial^k\partial^l h_{kl}^{(0)} - \partial^k\partial_k h_{(0)\,l}^{\,l})\eta^{ij} + \mathcal{O}(h^2)\,.$$

By the Wick-rotation of the boundary metric by $\tau = -it$ and introducing the complex coordinates as $z = x + i\tau$, one obtains the holomorphic linearized expression of stress tensor as

$$8\pi G\,T_{zz} = h_{zz}^{(2)} \,, \qquad 8\pi G\,T_{\bar{z}\bar{z}} = h_{\bar{z}\bar{z}}^{(2)} \,, \qquad 8\pi G\,T_{z\bar{z}} = h_{z\bar{z}}^{(2)} + \frac{1}{2}\Big(\partial^2 h_{\bar{z}\bar{z}}^{(0)} + \bar{\partial}^2 h_{zz}^{(0)} - 2\partial\bar{\partial}h_{z\bar{z}}^{(0)}\Big)\,.$$

To represent (on-shell) stress tensor in terms of boundary values, one needs to solve EOMs in terms of boundary values. More explicitly, one needs to solve $h_{(2)}$ in terms of $h_{(0)}$, which can be accomplished by linearizing EOMs. Note that ηi-components of linearized EOMs (from the 2nd order part of Fefferman-Graham expansion) is given by

$$\bar{\partial}h_{zz}^{(2)} = \partial h_{z\bar{z}}^{(2)} \,, \qquad \partial h_{\bar{z}\bar{z}}^{(2)} = \bar{\partial}h_{z\bar{z}}^{(2)} \,, \tag{29}$$

and $\eta\eta$-component of linearized EOMs is by

$$h_{z\bar{z}}^{(2)} = -\frac{1}{2}\Big(\partial^2 h_{\bar{z}\bar{z}}^{(0)} + \bar{\partial}^2 h_{zz}^{(0)} - 2\partial\bar{\partial}h_{z\bar{z}}^{(0)}\Big)\,, \tag{30}$$

which can be recognized as the linearization of the equation $R_{(0)} + 2\text{Tr}\,g_{(2)} = 0$.

Through the above linearized EOMs, one obtains linearized on-shell stress tensor in terms of boundary values as

$$T_{zz} = \frac{1}{8\pi G}\frac{\partial}{\bar{\partial}}h^{(2)}_{z\bar{z}} = -\frac{1}{16\pi G}\frac{\partial}{\bar{\partial}}\left(\partial^2 h^{(0)}_{\bar{z}\bar{z}} + \bar{\partial}^2 h^{(0)}_{zz} - 2\partial\bar{\partial}h^{(0)}_{z\bar{z}}\right), \qquad T_{z\bar{z}} = 0. \qquad (31)$$

Note that the first term in T_{zz} is non-local expression while the other two are local ones. The standard AdS/CFT dictionary dictates us the identification $T = \langle T\rangle_{CFT}$ and the prescription for correlators containing stress tensor as

$$\langle T^{ij}\cdots\rangle = \frac{2}{\sqrt{-\det\eta}}\frac{\delta}{\delta h^{(0)}_{ij}}\langle\cdots\rangle,$$

one obtains

$$\langle T_{zz}(z)T_{zz}(0)\rangle = \frac{3}{2G}\frac{1}{8\pi^2 z^4} \equiv \frac{c/2}{4\pi^2 z^4}, \qquad (32)$$

up to irrelevant local expression. There are similar expressions for anti-holomorphic quantities and correlators between holomorphic and anti-holomorphic stress tensor vanish. One can read the central charge of dual CFT from this formula as $c = 3/2G$.

Now, let us return to our case in NMG. At first sight, one may guess that one should settle ambiguities in counter terms and should treat the special point of $\sigma = 2m^2 = 1$ separately from a generic parameter point. On the contrary, at the linearized level, the condition of divergence cancelation of Brown-York stress tensor leads to the following universal form of renormalized stress tensor (up to overall scale $\sigma + 1/2m^2$)

$$8\pi G\, T^{ij} = 2g^{ij}_{(2)} + R_{(0)}g^{ij}_{(0)} = 2h^{ij}_{(2)} + (\partial^k\partial^l h^{(0)}_{kl} - \partial^k\partial_k h^{\ l}_{(0)\ l})\eta^{ij} + \mathcal{O}(h^2), \qquad (33)$$

which also holds at the special point. The simple reason of this form is that the $g_{(1)} = h_{(1)}$ term appears quadratically in finite part of Brown-York stress tensor. In some sense linearization eliminates ambiguities or the details of counter terms are irrelevant as long as they cancel the divergence of Brown-York tensor.

It is interesting to recognize that the linearized expression of renormalized boundary stress tensor in this case is just twice of the pure Einstein gravity case up to the given order. Since linearized EOMs in NMG are given by the same form in Eqs. (29) and (30) as can be shown by the linearization of Fefferman-Graham expansion of EOMs given in the Appendix B, one obtains

$$\langle T_{zz}(z)T_{zz}(0)\rangle = \left(\sigma + \frac{1}{2m^2}\right)\frac{3}{2G}\frac{1}{8\pi^2 z^4}, \qquad \langle T_{zz}(z)T_{\bar{z}\bar{z}}(0)\rangle = 0, \qquad (34)$$

with similar expression for anti-holomorphic stress tensor. This gives us the central charge of dual CFT as

$$c = \left(\sigma + \frac{1}{2m^2}\right)\frac{3}{2G}, \qquad (35)$$

which holds even at the special point of $\sigma = 2m^2 = 1$. It is also interesting to observe that $g_{(1)}$ term does not play any role in the linearized analysis because it appears in higher order terms than linear one. In previous sections, the $g_{(1)}$ term has some effects in various expressions since we have considered non-linear Fefferman-Graham expansions. A quadratic $g_{(1)}$ term is also important

requisite for obtaining the correct mass and angular momentum of new type black holes through holographically renormalized boundary stress tensor as will be shown in the next section.

One may wonder how NMG, which is a higher derivative theory, can give us the same linearized EOMs with Einstein gravity. The answer is that the magic of Fefferman-Graham expansion is played and reduces the radial derivatives effectively in some sense. The similar phenomenon occurs in the TMG case [34], where it was shown that linearized EOMs in TMG can be solved with some integration constants. It turns out that the final relevant equations for correlators are identical with the linearization of Fefferman-Graham expanded EOMs. That is to say, Fefferman-Graham expansion commutes with the linearization.

Now, let us explain why the condition Eq. (25) should be imposed at the special point of $\sigma = 2m^2 = 1$. In this linearized approach the central charge is determined uniquely by the value in Eq. (35) just with the condition of divergence cancelation. To be consistent with this result, we should impose the condition (25). Without this another condition for counter terms, the central charges obtained by the trace anomaly of renormalized stress tensor would be different from those by stress tensor correlators. That is to say, the self-consistency in holographic renormalization governs possible counter terms. As will be shown in the next section, this another condition is sufficient to obtain correctly mass and angular momentum of relevant black holes, *i.e.* new type black holes.

5   Conserved Charges and Renormalized Stress Tensors

It is a subtle problem to define conserved charges in gravity theory especially with higher derivative terms. However, in NMG there are some progress to define mass and angular momentum of AdS black hole solutions. Mass and angular momentum of BTZ black holes can be easily understood in many ways. Those of new type black holes are now understood through several studies. Holographically renormalized stress tensor can also be used to define conserved charges for black holes. As an application of renormalized stress tensor, we will derive mass and angular momentum of BTZ and new type black holes, which are consistent with the previous results. As one can see by this computation, it turns out that any counter term consistent with holographic renormalization leads to the correct mass and angular momentum of black holes, which means that the remaining ambiguities in counter terms are unphysical. This is also understood from the fact that the expression (22) of the renormalized stress tensor is independent of unphysical ambiguities. Here, the consistency with holographic renormalization means that only one condition of Eq. (15) holds for the $g_{(1)} = 0$ case and three conditions of Eqs. (15), (16) and (25) do for the $g_{(1)} \neq 0$ case.

It has been known that the mass and the angular momentum of black holes may be defined in terms of renormalized stress tensor as

$$M = \int_0^{2\pi} d\phi \ T^{tt}, \qquad J = \int_0^{2\pi} d\phi \ T^{t\phi}.$$

As the case of AdS black holes corresponding to the case of $g_{(1)} = 0$, let us consider the following

Schwarzschild form of BTZ black holes

$$ds^2 = \frac{r^2}{(r^2 - r_+^2)(r^2 - r_-^2)}dr^2 - \frac{(r^2 - r_+^2)(r^2 - r_-^2)}{r^2}dt^2 + r^2\Big(d\phi - \frac{r_+r_-}{r^2}dt\Big)^2,$$

which exist at any point of the parameter space. By the coordinate transformation

$$r^2 = e^{2\eta} + \frac{1}{2}(r_+^2 + r_-^2) + \frac{1}{16}e^{-2\eta}(r_+^2 - r_-^2)^2,$$

one obtains the Fefferman-Graham expanded form for the boundary metric $\gamma = e^{2\eta}g$ as

$$g_{ij}dx^i dx^j = -dt^2 + d\phi^2 + e^{-2\eta}\Big\{\frac{1}{2}(r_+^2 + r_-^2)(dt^2 + d\phi^2) - 2r_+r_-dtd\phi\Big\} + \mathcal{O}(e^{-4\eta}),$$

which shows us explicitly that $g_{(1)}$ vanishes in this case. Using renormalized stress tensor given in (17) for the $g_{(1)} = 0$ case, one obtains

$$M = \frac{r_+^2 + r_-^2}{8G}\Big(\sigma + \frac{1}{2m^2}\Big), \qquad J = \frac{r_+r_-}{4G}\Big(\sigma + \frac{1}{2m^2}\Big). \tag{36}$$

These are consistent with previous results by various other methods [11, 40, 43].

Now, let us consider new type black holes which exist at the special point $\sigma = 2m^2 = 1$. Since Fefferman-Graham expansion can be done completely for static new type black holes, rotating case will be dealt with separately. The metric of static new type black holes is given by

$$ds^2 = -(r^2 + br + c)dt^2 + \frac{dr^2}{(r^2 + br + c)} + r^2 d\phi^2, \tag{37}$$

with outer and inner horizons at $r_\pm = \frac{1}{2}(-b \pm \sqrt{b^2 - 4c})$. The mass of static new type black holes is identified as a conserved charge by [40]

$$M = \frac{b^2 - 4c}{16G}, \tag{38}$$

which is also justified by the dynamical approach [41] or AdS/CFT correspondence [43]. By the following coordinate transformation

$$r \equiv e^{\eta} - \frac{b}{2} + \frac{1}{4}e^{-\eta}\Big(\frac{b^2}{4} - c\Big), \tag{39}$$

one can see that the relevant part of the metric can be put in the Fefferman-Graham expanded form as

$$
\begin{aligned}
g_{ij}dx^i dx^j &= -\Big[1 - \frac{1}{4}e^{-2\eta}\Big(\frac{b^2}{4} - c\Big)\Big]^2 dt^2 + \Big[1 - \frac{b}{2}e^{-\eta} + \frac{1}{4}e^{-2\eta}\Big(\frac{b^2}{4} - c\Big)\Big]^2 d\phi^2 \\
&= -dt^2 + d\phi^2 - be^{-\eta}d\phi^2 + \frac{1}{2}e^{-2\eta}\Big[\Big(\frac{1}{4}b^2 - c\Big)dt^2 + \Big(\frac{3}{4}b^2 - c\Big)d\phi^2\Big] + \mathcal{O}(e^{-3\eta}).
\end{aligned}
$$

Using renormalized stress tensor relevant for the special point of $\sigma = 2m^2 = 1$ given in Eq. (24) equipped with Eq. (25), one obtains

$$8\pi G\, T^{tt} = 2g_{(2)}^{tt} + \mathrm{Tr}\, g_{(2)} - \frac{1}{4}(\mathrm{Tr}\, g_{(1)})^2 = \frac{1}{4}(b^2 - 4c). \tag{40}$$

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One can see that this gives the correct mass of static new type black holes [36, 40, 41] .

Now, let us consider the rotating new type black holes [12, 13].

$$ds^2 = -N^2(r)F^2(r)dt^2 + \frac{dr^2}{F^2(r)} + r^2\Big(d\phi + N^\phi(r)dt\Big)^2 \,, \tag{41}$$

where

$$N(r) \equiv 1 + \frac{b\alpha}{2H(r)} \,, \qquad N^\phi(r) \equiv \frac{\sqrt{\alpha(1-\alpha)}}{r^2}\Big(c + bH(r)\Big) \,,$$

$$F(r) \equiv \frac{H(r)}{r}\left[H^2(r) + b(1-\alpha)H(r) + \frac{b^2}{4}\alpha^2 + c(1-2\alpha)\right]^{1/2} \,,$$

$$H(r) \equiv \left[r^2 - \frac{b^2}{4}\alpha^2 + c\,\alpha\right]^{1/2} \,.$$

Note that α is the parameter for rotation ($0 \le \alpha \le 1/2$), which vanishes in the static case. In the following, one can see the explicit formula for the mass and the angular momentum of rotating new type black holes and so the relation of the parameter α with the mass and the angular momentum. The above metric can be set in the appropriate form for the Fefferman-Graham expansion at least up to relevant order by the following ansatz

$$r = e^\eta + \beta_1 + \beta_2 e^{-\eta} + \mathcal{O}(e^{-2\eta}) \,.$$

Using the coordinate transformation dictated by $dr/F(r) = d\eta$, one obtains

$$\beta_1 \equiv -\frac{b}{2}(1-\alpha) \,, \qquad \beta_2 \equiv \frac{1}{4}\left[\frac{b^2}{4}(1 - 2\alpha + 2\alpha^2) - c\right] \,.$$

Then, one can see that

$$g_{tt} = -1 - b\alpha\, e^{-\eta} - \left[b\beta_1 + \beta_1^2 + 2\beta_2 + c\right]e^{-2\eta} + \mathcal{O}(e^{-3\eta}) \,,$$

$$g_{\phi\phi} = 1 - b(1-\alpha)\, e^{-\eta} + (\beta_1^2 + 2\beta_2)e^{-2\eta} + \mathcal{O}(e^{-3\eta}) \,,$$

$$g_{t\phi} = b\sqrt{\alpha(1-\alpha)}\, e^{-\eta} + \sqrt{\alpha(1-\alpha)}(b\beta_1 + c)e^{-2\eta} + \mathcal{O}(e^{-3\eta}) \,,$$

which gives the correct mass and angular momentum of rotating new type black holes irrespective of the choice of counter terms with which the three conditions should be satisfied. Explicitly, one obtains

$$8\pi G\, T^{tt} = 2g_{(2)}^{tt} - g_{(1)}^{2\,tt} + \frac{1}{2}(\mathrm{Tr}\, g_{(1)})g_{(1)}^{tt} + \mathrm{Tr}\, g_{(2)} - \frac{1}{4}(\mathrm{Tr}\, g_{(1)})^2 = \frac{b^2}{4} - c \,, \tag{42}$$

$$8\pi G\, T^{t\phi} = 2g_{(2)}^{t\phi} - g_{(1)}^{2\,t\phi} + \frac{1}{2}(\mathrm{Tr}\, g_{(1)})g_{(1)}^{t\phi} = \Big(\frac{b^2}{4} - c\Big)2\sqrt{\alpha(1-\alpha)} \,. \tag{43}$$

Therefore, one can see that the mass and the ratio of the mass and the angular momentum is given respectively by

$$M = \frac{1}{16G}(b^2 - 4c) \,, \qquad \frac{J}{M} = 2\sqrt{\alpha(1-\alpha)} \,. \tag{44}$$

One may note that new type black holes become BTZ ones when the black hole parameter b vanishes and in this case the mass and the angular momentum of new type black holes also become those of BTZ black holes. The lesson of the above computation is that there is one parameter family of consistent counter terms and it leads to the correct mass and angular momentum of rotating new type black holes as well as central charges.

6 Conclusion

In this paper we have adopted Fefferman-Graham expansion in NMG and studied holographic renormalization. We have particularly focused on the special parameter point given by $\sigma = 2m^2 = 1$. At this special point the fall-off boundary condition is weaker than Brown-Henneaux one and so it needs to be analyzed separately. After obtaining renormalized stress tensor in generic Fefferman-Graham expansion, we obtain the central charge of dual CFT and then confirm mass and angular momentum of (rotating) new type black holes in this formalism. We have also solved the linearized EOMs in Fefferman-Graham expansion and then obtained correlators of renormalized boundary stress tensor. The central charge of dual CFT can also be read from these correlators.

To obtain renormalized stress tensor, we have considered generic counter terms and showed that there are some ambiguities in their construction, while physical quantities are innocent of these ambiguities. This is a reminiscent of renormalization scheme independence in field theories. However, it is not clear at this stage that this should really be understood as corresponding to scheme independence. The previous prescription for counter terms can be understood as minimal choices of those in our perspective.

Some comments are in order. At a generic point in the parameter space of the NMG Lagrangian, our expansion with $g_{(1)} = 0$ is a rather generic one and consistent with the fall-off conditions of the pure Einstein case (See Appendices B and C). In this case ambiguities in counter terms are rather large, while these ambiguities are unphysical. At the special point in our interest, our expansion is also consistent as shown in [12][36] and it satisfies a weaker fall-off condition than the Brown-Henneaux one. Our results show that this fall-off condition is consistent with holographic renormalization with ambiguities in counter terms. In fact, those ambiguities are shown to be physically irrelevant.

Now, it is useful to comment on our results in minds for the generality of our Fefferman-Graham expansion. Since we are dealing with a higher curvature or derivative theory, there is no guarantee that our expansion form is the most generic one, while the generic expansion in the pure Einstein case was derived rigorously in [44]. Our observation of the ambiguities in counter terms may be originated from the lack of generality of the Fefferman-Graham expansion. With the most generic expansion, it might be possible to determine the counter terms completely. Explicitly, when a more generic expansion is considered there may be other power law or 'log' terms leading to new divergence structure. To render finite new divergence structure, additional information about counter terms may be obtained and then the counter terms may be determined completely. However, even with the lack of the generality proof for the expansion our results are consistent and can be interpreted as the consistency of holographic renormalization with the restricted form of Fefferman-Graham expansion, which is reminiscent of the consistency claim of chiral gravity without 'log' modes. In other words, holographic renormalization for higher derivative theories has no inconsistency with a certain truncation.

Notably, it has been known that at the special point in our interest a more generic expansion is possible. That is to say, there may be the so-called 'log' terms, which is forbidden in a generic

point by equations of motions, in the form of

$$\gamma_{ij} = e^{2\eta}g_{ij}^{(0)} + e^{\eta}g_{ij}^{(1)} + \eta\,e^{\eta}b_{ij}^{(1)} + g_{ij}^{(2)} + e^{-\eta}g_{ij}^{(3)} + e^{-2\eta}g_{ij}^{(4)} + \mathcal{O}(e^{-3\eta})\,.$$

This 'log' fall-off condition is also shown to be consistent one [12][39]. With this more generic expansion, one may also study holographic renormalization and determine counter terms completely. However, it should be noted that this expansion satisfies different fall-off conditions and needs to be treated separately. One may also note that another 'log' term, $\eta b_{(2)}$, exists at the critical point which is different from the above 'log' term $\eta e^{\eta}b_{(1)}$, which needs to be investigated independently, as well.

Though there are some ambiguities in the choice of counter terms in our approach, it will be very interesting to investigate the possibility of determining counter terms completely at any point of parameter space through more generic, yet unknown, Fefferman-Graham expansion and/or other formalism. There have been some approaches to determine counter terms uniquely in the pure Einstein or Gauss-Bonnet gravity [52][53][54][55]. It will be very interesting to elaborate more on these methods for higher derivative gravity theories like NMG.

At the linearized analysis the existence of $g_{(1)}$ at the special point of $\sigma = 2m^2 = 1$ does not have a crucial role, since it does not lead to modification to EOMs of $h_{(2)}$ compared to the pure Einstein case. This situation is very similar to the case of log modes at the critical point. At the critical point, the existence of log modes leads to $AdS/LCFT$ correspondence, which is related to the introduction of new boundary source terms in the scheme of holographic renormalization, though the log fall-off behavior of bulk modes can be analyzed without their boundary source terms. Since these new boundary terms break asymptotic AdS properties, these are set to be zero at the end of computation, while their existence leads to new anomaly in LCFT, known as b central charge. It may be straightforward to apply such procedure to our case. However, it is unclear what is the dual interpretation of these new source terms in our case, since it is unclear which CFT corresponds to this case. It would be very interesting to see whether there is dual CFT with the inclusion of source terms for $g_{(k)}$ part of odd k in the bulk metric modes.

Though we have shown that Fefferman-Graham expansion reduces the linearized EOMs in NMG to those in the Einstein gravity case, it will be interesting to verify this statement by direct computation from linearized EOMs. By solving linearized EOMs with higher derivatives, one encounters several integration constants. After all, these constants should not play significant roles to obtain correlators for asymptotic AdS space as in TMG [34]. Another interesting problem is to apply holographic renormalization to extended NMGs. Since it was shown that new type black holes exist in R^3-NMG [43], it is clear that special points also exist in R^3-NMG. Since the parameter space is also extended in extended NMG, it is expected that there are several special subspaces. Therefore, it will be interesting to study more extensively the parameter space and see the role of holographic renormalization in these theories.

Acknowledgements

A Useful Formulae

In this appendix, we present various useful formulae in Fefferman-Graham coordinates for various geometrical quantities such as metric, extrinsic curvature and Ricci tensor. These formulae are used in the main text in order to compute Brown-York stress tensor and renormalized one.

The Fefferman-Graham expansion of the metric given in (6) leads to

$$
\begin{aligned}
\gamma^i_j &= g^i_{(0)j} = \eta^i_j\,, \\
\gamma^{ij} &= e^{-2\eta} g^{ij}_{(0)} - e^{-3\eta} g^{ij}_{(1)} - e^{-4\eta}\left(g^{ij}_{(2)} - (g^2_{(1)})^{ij}\right) \\
&\quad - e^{-5\eta}\left(g^{ij}_{(3)} - (g_{(1)}g_{(2)})^{ij} - (g_{(2)}g_{(1)})^{ij} + (g^3_{(1)})^{ij}\right) + \mathcal{O}(e^{-6\eta}) \\
\sqrt{-\gamma} &= e^{2\eta}\sqrt{-g_{(0)}}\left\{1 + \frac{1}{2}e^{-\eta}\mathrm{Tr}\,g_{(1)} + \frac{1}{2}e^{-2\eta}\left[\mathrm{Tr}\,g_{(2)} - \frac{1}{2}\mathrm{Tr}\,g^2_{(1)} + \frac{1}{4}(\mathrm{Tr}\,g_{(1)})^2\right]\right. \\
&\quad + \frac{1}{2}e^{-3\eta}\left[\mathrm{Tr}\,g_{(3)} - \mathrm{Tr}(g_{(1)}g_{(2)}) + \frac{1}{3}\mathrm{Tr}\,g^3_{(1)} + \frac{1}{2}\left(\mathrm{Tr}\,g_{(2)} - \frac{1}{2}\mathrm{Tr}\,g^2_{(1)} + \frac{1}{12}(\mathrm{Tr}\,g_{(1)})^2\right)\mathrm{Tr}\,g_{(1)}\right] \\
&\quad \left. + \mathcal{O}(e^{-4\eta})\right\}\,,
\end{aligned}
$$

where in the right hand side of equalities all indices are raised or lowered by $g^{(0)}$ not γ and Tr denotes the contraction with $g_{(0)}$. Christoffel symbols for the boundary metric are given by

$$
\begin{aligned}
\Gamma^k_{ij} &= \Gamma_{(0)\,ij}^{k} + \frac{1}{2}e^{-\eta}\,g^{kl}_{(0)}\left(\bar{\nabla}_i g^{(1)}_{lj} + \bar{\nabla}_j g^{(1)}_{il} - \bar{\nabla}_l g^{(1)}_{ij}\right) \\
&\quad + \frac{1}{2}e^{-2\eta}\left[g^{kl}_{(0)}\left(\bar{\nabla}_i g^{(2)}_{lj} + \bar{\nabla}_j g^{(2)}_{il} - \bar{\nabla}_l g^{(2)}_{ij}\right) - g^{kl}_{(1)}\left(\bar{\nabla}_i g^{(1)}_{lj} + \bar{\nabla}_j g^{(1)}_{il} - \bar{\nabla}_l g^{(1)}_{ij}\right)\right] + \mathcal{O}(e^{-3\eta}) \\
&\equiv \Gamma_{(0)\,ij}^{k} + e^{-\eta}\,\Gamma_{(1)\,ij}^{k} + e^{-2\eta}\,\Gamma_{(2)\,ij}^{k} + \mathcal{O}(e^{-3\eta})\,. \qquad\qquad\text{(A.1)}
\end{aligned}
$$

where $\bar{\nabla}$ denotes the covariant derivative with respect to $g_{(0)}$.

The Fefferman-Graham expansion of extrinsic curvature tensor is given by

$$
\begin{aligned}
K_{ij} &= e^{2\eta} g^{(0)}_{ij} + \frac{1}{2}e^{\eta} g^{(1)}_{ij} - \frac{1}{2}e^{-\eta} g^{(3)}_{ij} - e^{-2\eta} g^{(4)}_{ij} + \mathcal{O}(e^{-3\eta})\,, \\
K^i_j &= \eta^i_j - \frac{1}{2}e^{-\eta} g^i_{(1)j} - e^{-2\eta}\left[g^i_{(2)j} - \frac{1}{2}(g^2_{(1)})^i_j\right] \\
&\quad - e^{-3\eta}\left[\frac{3}{2}g^i_{(3)\,j} - (g_{(1)}g_{(2)})^i_j - \frac{1}{2}(g_{(2)}g_{(1)})^i_j + \frac{1}{2}(g^3_{(1)})^i_j\right] \\
&\quad - e^{-4\eta}\left[2g^i_{(4)\,j} - \frac{3}{2}(g_{(1)}g_{(3)})^i_j - \frac{1}{2}(g_{(3)}g_{(1)})^i_j - (g^2_{(2)})^i_j + (g^2_{(1)}g_{(2)})^i_j\right. \\
&\quad \left. + \frac{1}{2}(g_{(1)}g_{(2)}g_{(1)})^i_j + \frac{1}{2}(g_{(2)}g_{(1)})^i_j - \frac{1}{2}(g^4_{(1)})^i_j\right] + \mathcal{O}(e^{-4\eta})\,,
\end{aligned}
$$

which leads to

$$
\begin{aligned}
\sqrt{-\gamma}\left[(K-1)\gamma^{ij} - K^{ij}\right] &= \sqrt{-g_{(0)}}\left[\frac{1}{2}e^{-\eta}\left\{g^{ij}_{(1)} - (\mathrm{Tr}\,g_{(1)})g^{ij}_{(0)}\right\} + e^{-2\eta}\left\{g^{ij}_{(2)} - (\mathrm{Tr}\,g_{(2)})g^{ij}_{(0)}\right.\right. \\
&\quad \left.\left. - g^{ik}_{(1)}g^{j}_{(1)k} + \frac{3}{4}(\mathrm{Tr}\,g_{(1)})g^{ij}_{(1)} + \frac{1}{2}\left(\mathrm{Tr}\,(g_{(1)})^2 - \frac{1}{2}(\mathrm{Tr}\,g_{(1)})^2\right)g^{ij}_{(0)}\right\} + \cdots\right]\,.
\end{aligned}
$$

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Fefferman-Graham expansion for auxiliary fields $f^{\mu\nu}$'s from (13) are given by

$$
\begin{aligned}
m^2 f^{ij} &= -e^{-2\eta} g_{(0)}^{ij} + 2e^{-3\eta}\, g_{(1)}^{ij} + e^{-4\eta}\left[g_{(2)}^{ij} - 2g_{(1)\,k}^{i} g_{(1)}^{kj} - \frac{1}{2}(\operatorname{Tr} g_{(1)})\, g_{(1)}^{ij} \right. \\
&\qquad \left. + 2\left(R_{(0)}^{ij} - \frac{1}{4}R_{(0)}g_{(0)}^{ij}\right) + \left(\operatorname{Tr} g_{(2)} - \frac{3}{8}\operatorname{Tr} g_{(1)}^{2} + \frac{1}{8}(\operatorname{Tr} g_{(1)})^2\right) g_{(0)}^{ij} \right] + \mathcal{O}(e^{-5\eta}), \\
m^2 f^{i}_{\ j} &= -\eta^{i}_{\ j} + e^{-\eta}\, g_{(1)\,j}^{\ i} + \frac{1}{2}\, e^{-2\eta}\left[\left(R_{(0)} + 2\operatorname{Tr} g_{(2)} - \frac{3}{4}\operatorname{Tr} g_{(1)}^{2} + \frac{1}{4}(\operatorname{Tr} g_{(1)})^2\right)\eta^{i}_{\ j} \right. \\
&\qquad \left. - (\operatorname{Tr} g_{(1)})\, g_{(1)\,j}^{\ i} \right] + \mathcal{O}(e^{-3\eta}), \\
m^2 h^{i} &= e^{-3\eta}\left[\bar{\nabla}^{i}\operatorname{Tr} g_{(1)} - \bar{\nabla}_{j} g_{(1)}^{ji} \right] + 2e^{-4\eta}\left[\bar{\nabla}^{i}\operatorname{Tr} g_{(2)} - \bar{\nabla}_{j} g_{(2)}^{ji} \right. \\
&\qquad \left. + g_{(1)j}^{i}\bar{\nabla}_{k} g_{(1)}^{kj} + \frac{1}{2} g_{(1)k}^{j}\bar{\nabla}_{j} g_{(1)}^{ik} - \frac{3}{4} g_{(1)}^{kl}\bar{\nabla}^{i} g_{kl}^{(1)} - \frac{3}{4} g_{(1)}^{ij}\bar{\nabla}_{j}\operatorname{Tr} g_{(1)} \right] + \mathcal{O}(e^{-5\eta}), \\
m^2 s &= -1 - \frac{1}{2} e^{-2\eta}\left[R_{(0)} + 2\operatorname{Tr} g_{(2)} - \frac{1}{4}\operatorname{Tr} g_{(1)}^{2} - \frac{1}{4}(\operatorname{Tr} g_{(1)})^2 \right] + \cdots . \tag{A.2}
\end{aligned}
$$

Recall that $\hat{f} \equiv \gamma^{ij} f_{ij}$, then

$$
m^2 \hat{f} = -2 + e^{-\eta}\operatorname{Tr} g_{(1)} + e^{-2\eta}\left[R_{(0)} + 2\operatorname{Tr} g_{(2)} - \frac{3}{4}\operatorname{Tr} g_{(1)}^{2} - \frac{1}{4}(\operatorname{Tr} g_{(1)})^2 \right] + \mathcal{O}(e^{-3\eta}).
$$

Note that the original f is given by $f = \hat{f} + s$ as

$$
m^2 f = \frac{1}{2}R = -3 + e^{-\eta}\operatorname{Tr} g_{(1)} + \frac{1}{2}e^{-2\eta}\left[R_{(0)} + 2\operatorname{Tr} g_{(2)} - \frac{5}{4}\operatorname{Tr} g_{(1)}^{2} - \frac{1}{4}(\operatorname{Tr} g_{(1)})^2 \right] + \mathcal{O}(e^{-3\eta}).
$$

Some useful formulae for the computation of Brown-York tensor are

$$
\begin{aligned}
-\nabla^{(i} h^{j)} + \gamma^{ij}\nabla_{k} h^{k} &= \mathcal{O}(e^{-5\eta}) \\
\frac{1}{2}D_{\eta} f^{ij} + K_{k}^{(i} f^{j)k} - \frac{1}{2}\gamma^{ij} D_{\eta} f &= -\frac{1}{2m^2}e^{-3\eta}\left(g_{(1)}^{ij} - \operatorname{Tr} g_{(1)} g_{(0)}^{ij} \right) + \frac{1}{2m^2}e^{-4\eta}\left[(g_{(1)}^{2})^{ij} + R_{(0)}g_{(0)}^{ij} \right. \\
&\qquad \left. + \left(2\operatorname{Tr} g_{(2)} - \frac{3}{4}(\operatorname{Tr} g_{(1)})^2 - \frac{3}{4}\operatorname{Tr} g_{(1)}^{2} \right) g_{(0)}^{ij} \right] + \mathcal{O}(e^{-5\eta})
\end{aligned}
$$

Useful formulae for counter terms computation are

$$
\begin{aligned}
m^4 \hat{f}^2 &= 4 - 4e^{-\eta}\operatorname{Tr} g_{(1)} - 4e^{-2\eta}\left[R_{(0)} + 2\operatorname{Tr} g_{(2)} - \frac{3}{4}\operatorname{Tr} g_{(1)}^{2} - \frac{1}{2}(\operatorname{Tr} g_{(1)})^2 \right] + \mathcal{O}(e^{-3\eta}) \\
m^4 f_{kl} f^{kl} &= 2 - 2e^{-\eta}\operatorname{Tr} g_{(1)} - 2e^{-2\eta}\left[R_{(0)} + 2\operatorname{Tr} g_{(2)} - \frac{5}{4}\operatorname{Tr} g_{(1)}^{2} - \frac{1}{4}(\operatorname{Tr} g_{(1)})^2 \right] + \mathcal{O}(e^{-3\eta})
\end{aligned}
$$

Fefferman-Graham expansion for Ricci tensor and Ricci scalar is given by

$$
\begin{aligned}
R^i_j &= -2\eta^i_j + \frac{1}{2}e^{-\eta}\Big[g^i_{(1)j} + (\mathrm{Tr}\, g_{(1)})\eta^i_j\Big] \\
&\quad + e^{-2\eta}\Big[\frac{1}{2}\Big(R_{(0)} + 2\mathrm{Tr}\, g_{(2)} - \mathrm{Tr}\, g^2_{(1)}\Big)\eta^i_j - \frac{1}{4}(\mathrm{Tr}\, g_{(1)})g^i_{(1)j}\Big] \\
&\quad + \frac{1}{2}e^{-3\eta}\Big[-R_{(0)}g_{(1)}{}^i{}_j + 2X^i_j + \Big(3\mathrm{Tr}\, g_{(3)} - 3\mathrm{Tr}(g_{(1)}g_{(2)}) + \mathrm{Tr}\, g^3_{(1)}\Big)\eta^i_j \\
&\qquad - \Big(3g_{(3)}{}^i{}_j - 2(g_{(1)}g_{(2)})^i_j - (g_{(2)}g_{(1)})^i_j + (g^3_{(1)})^i_j\Big) \\
&\qquad - \Big(g_{(2)}{}^i{}_j - \frac{1}{2}(g^2_{(1)})^i_j\Big)\mathrm{Tr}\, g_{(1)} - \Big(\mathrm{Tr}\, g_{(2)} - \frac{1}{2}\mathrm{Tr}\, g^2_{(1)}\Big)g_{(1)}{}^i{}_j\Big] \\
&\quad + e^{-4\eta}\Big[Y^i_j - (g_{(1)}X)^i_j - \frac{1}{2}R_{(0)}\Big(g_{(2)}{}^i{}_j - (g^2_{(1)})^i_j\Big) \\
&\qquad - 4g_{(4)}{}^i{}_j + 3(g_{(1)}g_{(3)})^i_j + (g_{(3)}g_{(1)})^i_j + 2(g^2_{(2)})^i_j - 2(g^2_{(1)}g_{(2)})^i_J - (g_{(1)}g_{(2)}g_{(1)})^i_j \\
&\qquad - (g_{(2)}g^2_{(1)})^i_j + (g^4_{(1)})^i_j - \frac{1}{4}\Big(3g_{(3)} - 2g_{(1)}g_{(2)} - g_{(2)}g_{(1)} + g^3_{(1)}\Big)^i{}_j \mathrm{Tr}\, g_{(1)}\Big] \\
&\qquad - \Big(\mathrm{Tr}\, g_{(2)} - \frac{1}{2}\mathrm{Tr}\, g^2_{(1)}\Big)\Big(g_{(2)} - \frac{1}{2}g^2_{(1)}\Big)^i_j - \frac{1}{4}\Big(3\mathrm{Tr}\, g_{(3)} - 3\mathrm{Tr}(g_{(1)}g_{(2)}) + \mathrm{Tr}\, g^3_{(1)}\Big)g_{(1)}{}^i{}_j \\
&\qquad + \Big(2\mathrm{Tr}\, g_{(4)} - 2\mathrm{Tr}(g_{(1)}g_{(3)}) - \mathrm{Tr}\, g^2_{(2)} + 2\mathrm{Tr}(g^2_{(1)}g_{(2)}) - \frac{1}{2}\mathrm{Tr}\, g^4_{(1)}\Big)\eta^i_j\Big] \\
&\quad + \mathcal{O}(e^{-5\eta})\,,
\end{aligned}
$$

$$
\begin{aligned}
R^\eta{}_\eta &= -2 + \frac{1}{2}e^{-\eta}\,\mathrm{Tr}\, g_{(1)} - \frac{1}{4}e^{-2\eta}\,\mathrm{Tr}\, g^2_{(1)} - \frac{1}{2}e^{-3\eta}\Big[3\mathrm{Tr}\, g_{(3)} - \mathrm{Tr}(g_{(1)}g_{(2)})\Big] \\
&\quad - e^{-4\eta}\Big[4\mathrm{Tr}\, g_{(4)} - \frac{5}{2}\mathrm{Tr}(g_{(1)}g_{(3)}) - \mathrm{Tr}\, g^2_{(2)} + \frac{3}{2}\mathrm{Tr}(g^2_{(1)}g_{(2)}) - \frac{1}{4}\mathrm{Tr}\, g^4_{(1)}\Big] + \mathcal{O}(\eta^{-5\eta})\,,
\end{aligned}
$$

$$
\begin{aligned}
R^\eta{}_i &= \frac{e^{-\eta}}{2}\Big[\bar\nabla_i \mathrm{Tr} g_{(1)} - \bar\nabla_j g^j_{(1)\,i}\Big] + e^{-2\eta}\Big[\bar\nabla_i \mathrm{Tr} g_{(2)} - \bar\nabla_j g^j_{(2)\,i} \\
&\qquad + \frac{1}{2}g^{(1)}_{ij}\bar\nabla_k g^{jk}_{(1)} + \frac{1}{2}g^{jk}_{(1)}\bar\nabla_j g^{(1)}_{ki} - \frac{3}{4}g^{kl}_{(1)}\bar\nabla_i g^{(1)}_{kl} - \frac{1}{4}g^{(1)j}_i \bar\nabla_j \mathrm{Tr}\, g_{(1)}\Big] + \mathcal{O}(\eta^{-3\eta})\,,
\end{aligned}
$$

$$
\begin{aligned}
R &= -6 + 2e^{-\eta}\mathrm{Tr}\, g_{(1)} + e^{-2\eta}\Big[R_{(0)} + 2\mathrm{Tr}\, g_{(2)} - \frac{5}{4}\mathrm{Tr}\, g^2_{(1)} - \frac{1}{4}(\mathrm{Tr}\, g_{(1)})^2\Big] \\
&\quad + e^{-3\eta}\Big[\mathrm{Tr}\, X - \mathrm{Tr}(g_{(1)}g_{(2)}) + \frac{1}{2}\mathrm{Tr}\, g^3_{(1)} - \frac{1}{2}\Big(R_{(0)} + 2\mathrm{Tr}\, g_{(2)} - \mathrm{Tr}\, g^2_{(1)}\Big)\mathrm{Tr}\, g_{(1)}\Big] \\
&\quad + e^{-4\eta}\Big[\mathrm{Tr}\, Y - \mathrm{Tr}(g_{(1)}X) - 4\mathrm{Tr}\, g_{(4)} + \mathrm{Tr}\, g^2_{(2)} + \frac{5}{2}\mathrm{Tr}(g_{(1)}g_{(3)}) - \frac{3}{2}\mathrm{Tr}(g^2_{(1)}g_{(2)}) + \frac{1}{4}\mathrm{Tr}\, g^4_{(1)} \\
&\qquad - \frac{1}{2}\Big(R_{(0)} + 2\mathrm{Tr}\, g_{(2)}\Big)\Big(\mathrm{Tr}\, g_{(2)} - \mathrm{Tr}\, g^2_{(1)}\Big) - \frac{1}{4}(\mathrm{Tr}\, g^2_{(1)})^2 \\
&\qquad - \Big(\frac{3}{2}\mathrm{Tr}\, g_{(3)} - \frac{3}{2}\mathrm{Tr}(g_{(1)}g_{(2)}) + \frac{1}{2}\mathrm{Tr}\, g^3_{(1)}\Big)\mathrm{Tr}\, g_{(1)}\Big] + \mathcal{O}(e^{-5\eta})\,,
\end{aligned}
$$

where X, Y are defined in terms of (A.1) by

$$
\begin{aligned}
X^i_j &\equiv \frac{1}{2}\Big(\bar\nabla^k \bar\nabla^i g_{(1)\,jk} + \bar\nabla^k \bar\nabla_j g_{(1)}{}^i{}_k - \bar\nabla^i \bar\nabla_j \mathrm{Tr}\, g_{(1)} - \bar\nabla^2 g_{(1)}{}^i{}_j\Big)\,, \\
Y^i_j &\equiv g^{ik}_{(0)}\Big(\bar\nabla_l \Gamma^l_{(2)\,jk} - \bar\nabla_j \Gamma^l_{(2)\,lk} + \Gamma^l_{(1)\,lm}\Gamma^m_{(1)\,jk} - \Gamma^l_{(1)\,jm}\Gamma^m_{(1)\,lk}\Big)\,.
\end{aligned}
\tag{A.3}
$$

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B   Equations of Motions in Fefferman-Graham Coordinates

In this appendix we present EOMs in the Fefferman-Graham expanded form, through which the relevant linearized EOMs can be obtained. Before presenting the expression for $\mathcal{E}^\mu_\nu$, let us consider the Fefferman-Graham expansion of the $\mathcal{K}^\mu_\nu$ term, which are given by

$$
\begin{aligned}
\mathcal{K}^\eta_\eta &= -\frac{1}{2} + \frac{1}{2}e^{-\eta}\,\mathrm{Tr}\,g_{(1)} + e^{-2\eta}\left[\frac{1}{2}R_{(0)} + \mathrm{Tr}\,g_{(2)} - \frac{1}{8}\mathrm{Tr}\,g^2_{(1)} - \frac{3}{8}\left(\mathrm{Tr}\,g_{(1)}\right)^2\right] \\
&\quad + e^{-3\eta}\left[\frac{1}{2}\left(\bar{\nabla}^i\bar{\nabla}^j g_{(1)\,ij} - \bar{\nabla}^2\mathrm{Tr}\,g_{(1)}\right) - \frac{1}{2}R_{(0)}\,\mathrm{Tr}\,g_{(1)} - \mathrm{Tr}\left(g_{(1)}\,g_{(2)}\right) - \frac{9}{16}\mathrm{Tr}\,g_{(1)}\,\mathrm{Tr}\,g^2_{(1)}\right. \\
&\qquad\left. - \mathrm{Tr}\,g_{(1)}\,\mathrm{Tr}\,g_{(2)} + \frac{3}{2}\mathrm{Tr}\,g_{(3)} + \frac{3}{4}\mathrm{Tr}\,g^3_{(1)} + \frac{7}{16}(\mathrm{Tr}\,g_{(1)})^3\right] + \mathcal{O}(e^{-4\eta}), \quad\quad\text{(A.4)}
\end{aligned}
$$

$$
\mathcal{K}^\eta_i = -\frac{e^{-\eta}}{2}\left[\bar{\nabla}_i\mathrm{Tr}g_{(1)} - \bar{\nabla}_j g^j_{(1)\,i}\right] + e^{-2\eta}\left[\bar{\nabla}_i\mathrm{Tr}g_{(2)} - \bar{\nabla}_j g^j_{(2)\,i} + \mathcal{O}(g^2_{(1)})\right] + \mathcal{O}(e^{-3\eta}), \quad\text{(A.5)}
$$

$$
\begin{aligned}
\mathcal{K}^i_j &= -\frac{1}{2}\eta^i_j + e^{-\eta}\left[\frac{1}{2}\left(\mathrm{Tr}\,g_{(1)}\right)\eta^i_j - \frac{1}{2}g^i_{(1)j}\right] \\
&\quad + e^{-2\eta}\left[R^i_{(0)j} + \left\{-\frac{1}{2}R_{(0)} - \frac{1}{8}\left((\mathrm{Tr}\,g_{(1)})^2 + \mathrm{Tr}\,g^2_{(1)}\right)\right\}\eta^i_j + \frac{1}{4}(\mathrm{Tr}\,g_{(1)})g^i_{(1)j}\right] \\
&\quad + \mathcal{O}(e^{-3\eta}), \quad\quad\text{(A.6)}
\end{aligned}
$$

where we have kept the necessary terms up to the relevant orders.

Using the above $\mathcal{K}$-tensor expressions in conjunction with the Ricci tensor ones in the previous appendix, one obtains EOM expression as

$$
\begin{aligned}
\mathcal{E}^\eta_\eta &= \left(\sigma - \frac{1}{\ell^2} - \frac{1}{4m^2}\right) + e^{-\eta}\left(-\frac{\sigma}{2} + \frac{1}{4m^2}\right)\mathrm{Tr}\,g_{(1)} \\
&\quad + e^{-2\eta}\left[\left(-\frac{\sigma}{2} + \frac{1}{4m^2}\right)\left(R_{(0)} + 2\mathrm{Tr}\,g_{(2)}\right) + \left(\frac{3\sigma}{8} - \frac{1}{16m^2}\right)\mathrm{Tr}\,g^2_{(1)} + \left(\frac{\sigma}{8} - \frac{3}{16m^2}\right)(\mathrm{Tr}\,g_{(1)})^2\right] \\
&\quad + e^{-3\eta}\left[\left(-\frac{\sigma}{2} + \frac{1}{4m^2}\right)\left(\bar{\nabla}^i\bar{\nabla}^j g_{(1)\,ij} - \bar{\nabla}^2\mathrm{Tr}\,g_{(1)}\right) + \left(\frac{\sigma}{4} - \frac{1}{4m^2}\right)R_{(0)}\,\mathrm{Tr}\,g_{(1)}\right. \\
&\qquad + \left(\sigma - \frac{1}{2m^2}\right)\left(\mathrm{Tr}(g_{(1)}g_{(2)}) - \frac{3}{2}\mathrm{Tr}\,g_{(3)}\right) - \left(\frac{\sigma}{4} + \frac{9}{32m^2}\right)\mathrm{Tr}\,g_{(1)}\,\mathrm{Tr}\,g^2_{(1)} \\
&\qquad\left. + \left(\frac{\sigma}{2} - \frac{1}{2m^2}\right)\mathrm{Tr}\,g_{(1)}\,\mathrm{Tr}\,g_{(2)} - \left(\frac{\sigma}{4} - \frac{3}{8m^2}\right)\mathrm{Tr}\,g^3_{(1)} + \frac{7}{32m^2}(\mathrm{Tr}\,g_{(1)})^3\right] + \mathcal{O}(e^{-4\eta}), \text{(A.7)}
\end{aligned}
$$

$$
\mathcal{E}^\eta_j = e^{-\eta}\left[\left(\frac{\sigma}{2} - \frac{1}{4m^2}\right)\left(\bar{\nabla}_i\mathrm{Tr}g_{(1)} - \bar{\nabla}_j g^j_{(1)\,i}\right)\right] + e^{-2\eta}\left[\left(\sigma + \frac{1}{2m^2}\right)\left(\bar{\nabla}_i\mathrm{Tr}g_{(2)} - \bar{\nabla}_j g^j_{(2)\,i}\right) + \mathcal{O}(g^2_{(1)})\right]
$$
$$
+ \mathcal{O}(e^{-3\eta}), \quad\quad\text{(A.8)}
$$

$$
\begin{aligned}
\mathcal{E}^i_j &= \left(\sigma - \frac{1}{\ell^2} - \frac{1}{4m^2}\right)\eta^i_j + e^{-\eta}\left[\left(\frac{\sigma}{2} - \frac{1}{4m^2}\right)\left(g^i_{(1)j} - (\mathrm{Tr}\,g_{(1)})\eta^i_j\right)\right] \\
&\quad + e^{-2\eta}\left[\left(\sigma - \frac{1}{2m^2}\right)\left\{\frac{1}{8}\left((\mathrm{Tr}\,g_{(1)})^2 + \mathrm{Tr}\,g^2_{(1)}\right)\eta^i_j - \frac{1}{4}(\mathrm{Tr}\,g_{(1)})g^i_{(1)j}\right\}\right. \\
&\qquad\left. + \left(\sigma + \frac{1}{2m^2}\right)\left(R^i_{(0)j} - \frac{1}{2}R_{(0)}\eta^i_j\right)\right] + \mathcal{O}(e^{-3\eta}), \quad\quad\text{(A.9)}
\end{aligned}
$$

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where the two dimension identity $R^i_{(0)j} - \frac{1}{2} R_{(0)} \eta^i_j = 0$ is used.

These EOMs imply that $g_{(1)}$ term should vanish at the generic point in the parameter space except the special point $\sigma = 2m^2 = 1 = 2/\ell^2$ where the last equality is from Eq.(5). More concretely, the first term in $\mathcal{E}^\eta_\eta = \mathcal{E}^i_j = 0$ give us the condition $\sigma - 1/\ell^2 - 1/4m^2 = 0$, and the second term gives $g_{(1)} = 0$ or $\sigma = 2m^2$. Therefore, one can see that $g_{(1)} = 0$ except the special point $\sigma = 2m^2 = 1$. At a generic parameter point, the next order term from EOMs leads to

$$R_{(0)} + 2\mathrm{Tr}\, g_{(2)} = 0\,.$$

At the special point of $\sigma = 2m^2 = 1 = 2/\ell^2$, the $\eta\eta$-component of EOMs, $\mathcal{E}^\eta_\eta = 0$, gives the following two conditions

$$(\mathrm{Tr}\, g_{(1)})^2 = \mathrm{Tr}\, g^2_{(1)}\,,$$
$$\left(R_{(0)} + 2\mathrm{Tr}\, g_{(2)} - \frac{1}{2}(\mathrm{Tr}\, g_{(1)})^2\right) \mathrm{Tr}\, g_{(1)} = 0\,,$$

where the first condition comes from $e^{-2\eta}$ order and the second one from $e^{-3\eta}$ order using the relation $(\mathrm{Tr}\, g_{(1)})^3 = \mathrm{Tr}\, g^3_{(1)}$ derived by the first condition in two dimensions. Note that the ij-component of EOMs, $\mathcal{E}^i_j = 0$, does not give a non-trivial condition up to $e^{-2\eta}$ order, and the ηi-component of EOMs, $\mathcal{E}^\eta_i = 0$, also gives no condition up to $e^{-\eta}$ order.

Even at the special point $\sigma = 2m^2 = 1$, $g_{(1)}$ may vanish. In this case, the $\eta\eta$-component of EOMs gives no condition at the $e^{-3\eta}$ order. But one condition can be obtained at $e^{-4\eta}$ order. By computing the fully contracted EOM, this fact can be easily seen as follows. Since FG expansion of $\mathcal{K}$-term is given by

$$
\begin{aligned}
\mathcal{K} &= -\frac{3}{2} + \frac{1}{2}e^{-2\eta}\left[R_{(0)} + 2\mathrm{Tr}\, g_{(2)}\right] \\
&\quad + \frac{1}{2}e^{-4\eta}\left[-4\mathrm{Tr}\, g_{(4)} + \mathrm{Tr}\, g^2_{(2)} - \frac{1}{4}(R_{(0)} + 2\mathrm{Tr}\, g_{(2)})(R_{(0)} + 4\mathrm{Tr}\, g_{(2)})\right. \\
&\quad \left. + g^{ik}_{(0)}\left(\bar{\nabla}_l \Gamma^l_{(2)\,ik} - \bar{\nabla}_i \Gamma^l_{(2)\,lk}\right)\right] + \mathcal{O}(e^{-5\eta})\,,
\end{aligned}
$$

at these special point, fully contracted EOM expression is given by

$$\mathcal{E}^\mu_\mu = -\frac{1}{8}\, e^{-4\eta}\left(R_{(0)} + 2\mathrm{Tr}\, g_{(2)}\right)^2 + O(e^{-5\eta})\,. \tag{A.10}$$

Therefore when $g_{(1)} = 0$, regardless of the value of the parameter m^2 we have the following condition.

$$R_{(0)} + 2\mathrm{Tr}\, g_{(2)} = 0\,. \tag{A.11}$$

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C The Absence of 'Log' Term at a Generic Point

In this appendix, we show that there is no 'log' terms, which is proportial to η in our convention, at a generic point in the parameter space. Let us begin with the following Fefferman-Graham expansion with 'log' term,

$$\gamma_{ij} = e^{2\eta} g^{(0)}_{ij} + e^{\eta} g^{(1)}_{ij} + \eta e^{\eta} b^{(1)}_{ij} + g^{(2)}_{ij} + \eta b^{(2)}_{ij} + e^{-\eta} g^{(3)}_{ij} + e^{-2\eta} g^{(4)}_{ij} + \mathcal{O}(e^{-3\eta}),\qquad (A.12)$$

The fully contracted EOM is expressed in the form of

$$\mathcal{E}^{\mu}_{\mu} = 3\left(\sigma - \frac{1}{\ell^2} - \frac{1}{4m^2}\right) + e^{-\eta}\left[\left(-\sigma + \frac{1}{2m^2}\right)\left(\operatorname{Tr} g_{(1)} - \frac{1}{2}\operatorname{Tr} b_{(1)}\right)\right]$$

$$+\eta e^{-\eta}\left(-\sigma + \frac{1}{2m^2}\right)\operatorname{Tr} b_{(1)} + \mathcal{O}(e^{-2\eta}) = 0. \qquad (A.13)$$

At a generic point, one can see that $\operatorname{Tr} b_{(1)} = 0$. Since the ij component of EOM is given by

$$\mathcal{E}^{i}_{j} = \left(\sigma - \frac{1}{\ell^2} - \frac{1}{4m^2}\right)\eta^i_j + e^{-\eta}\left[\left(\frac{\sigma}{2} - \frac{1}{4m^2}\right)\left(g^i_{(1)j} - (\operatorname{Tr} g_{(1)})\,\eta^i_j\right)\right]$$

$$+\eta e^{-\eta}\left[\left(\frac{\sigma}{2} - \frac{1}{4m^2}\right)\left(b^i_{(1)j} - (\operatorname{Tr} b_{(1)})\,\eta^i_j\right)\right] + \mathcal{O}(e^{-2\eta}) = 0, \qquad (A.14)$$

one can insure that $b^i_{(1)j} = 0$ at a generic point.

Using this result, the second order of the fully contracted EOM can be written simply as

$$e^{-2\eta}\left[\left(-\frac{\sigma}{2} + \frac{1}{4m^2}\right)\left(R_{(0)} + 2\operatorname{Tr} g_{(2)} + \operatorname{Tr} b_{(2)}\right)\right]$$

$$+\eta e^{-2\eta}\left(-\sigma + \frac{1}{2m^2}\right)\operatorname{Tr} b_{(2)} + \mathcal{O}(e^{-3\eta}) = 0. \qquad (A.15)$$

From this, one can also see that $\operatorname{Tr} b_{(2)} = 0$ at a generic point. Since the second order of ij component of EOM is given by

$$e^{-2\eta}\left[\left(\sigma + \frac{1}{2m^2}\right)b^i_{(2)j} - \sigma(\operatorname{Tr} b_{(2)})\,\eta^i_j\right] + \mathcal{O}(e^{-3\eta}) = 0, \qquad (A.16)$$

one can also verify that $b^i_{(2)j} = 0$ at a generic point.

D The Computation of the On-shell Action Value

Though it is naturally expected that the divergence cancelation condition from the on-shell action value is identical with the one from stress tensor, the on-shell action value is presented in this appendix. Let us put the AdS boundary at the radius $\eta = \eta_\infty$ near the asymptotic infinity. This renders finite the on-shell action value, while the divergence of the on-shell action value appear by sending $\eta_\infty \to \infty$. Explicitly, the on-shell action value of NMG in Fefferman-Graham coordinates is given by

$$
\begin{aligned}
S_{NMG} &= \frac{1}{16\pi G} \int d^3x \sqrt{-\gamma} \Big[\sigma R + \frac{2}{\ell^2} + \frac{1}{m^2} \mathcal{K} \Big] \\
&= \frac{1}{16\pi G} \int d^2x \sqrt{-g_{(0)}} \Big(\sigma + \frac{1}{2m^2} \Big) \Big[-2e^{2\eta_\infty} + \eta_\infty R_{(0)} + \cdots \Big],
\end{aligned}
\tag{A.17}
$$

where we have kept the potentially divergent part and $\cdots$ denote the finite part. Note that the $g_{(1)}$ term doesn't appear in the divergent part. The above divergent part is completely same with the pure Einstein gravity case. That is to say, for the counter action $S_{c.t.}$ one needs just the boundary cosmological constant term and $(\eta/16\pi G) \int d^2x \sqrt{-\gamma} R[\gamma]$ term which is omitted in our paper since it is trivial under metric variation in two dimension.

It is straightforward to obtain the on-shell value of generalized GH terms as

$$
\begin{aligned}
S_{GH} &= \frac{\xi}{16\pi G} \int d^2x \sqrt{-\gamma} \Big[2\sigma K + \hat{f}^{ij} K_{ij} - \hat{f} K \Big] \\
&= \frac{\xi}{16\pi G} \int d^2x \sqrt{-g_{(0)}} \Big[4e^{2\eta_\infty} \Big(\sigma + \frac{1}{2m^2} \Big) + e^{\eta_\infty} \Big(\sigma - \frac{1}{2m^2} \Big) \operatorname{Tr} g_{(1)} \Big] + \cdots .
\end{aligned}
\tag{A.18}
$$

The on-shell value of counter terms is given by

$$
\begin{aligned}
S_{c.t.} &= \frac{1}{8\pi G} \int d^2x \sqrt{-\gamma} \, (A + B\hat{f} + C\hat{f}^2 + D f_{ij} f^{ij}) \\
&= \frac{1}{8\pi G} \int d^2x \sqrt{-g_{(0)}} \Big[e^{2\eta_\infty} \Big(A - \frac{2B}{m^2} + \frac{4C}{m^4} + \frac{2D}{m^4} \Big) + e^{\eta_\infty} \Big(\frac{1}{2}A - \frac{2C}{m^4} - \frac{D}{m^4} \Big) \operatorname{Tr} g_{(1)} \\
&\quad + \cdots \Big].
\end{aligned}
\tag{A.19}
$$

Now, one can ensure that the divergence cancelation condition is completely identical with stress tensor computation.

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